

Gravitomagnetic forces and quadrupole gravitational radiation from special relativity

P. Christillin
Dipartimento di Fisica,
Università di Pisa
I.N.F.N. Sezione di Pisa

and

L. Barattini
Università di Pisa

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Abstract

The mere principle of relativity and Lorentz transformations for the mass current predict, in close analogy to electromagnetism, the existence of gravitomagnetic fields. With the reasonable assumption of the non existence of a gravitomagnetic mass, a parameter free set of equations for “effective” vector gravitoelectromagnetism results. As a consequence gravity propagates at the speed of light in a flat Minkowski space and quadrupole gravitational radiation, consistent with energy balance, is predicted with the same value of GR.

1 Introduction

The elusive quest for the direct experimental detection of the gravitational quadrupole radiation

$$W_G = \frac{G}{45c^5} \left(\frac{d^3 Q}{dt^3} \right)^2 \quad (1)$$

where Q stands for the usual mass quadrupole, is generally considered to be a crucial test of Einstein’s General Relativity (GR) ([1]). As well known, so far, only an indirect evidence of such an effect has come from the observation of binary pulsars (PSR B1913+16 [2] and PSR J0737-3039A/B [3]). This has however strengthened the general belief of space-time curved by matter and of gravitational waves as “ripples which propagate in space time”. On the

other hand it has long been noticed that the corresponding expression for the electromagnetic quadrupole radiation

$$W_{el} = \frac{1}{4\pi\epsilon_0} \frac{1}{180c^5} \left(\frac{d^3Q}{dt^3} \right)^2 \quad (2)$$

(where this time Q represents the electric quadrupole connected to the previous one by the obvious replacement $q \rightarrow m$) is strikingly similar to the gravitational one. Given the obvious correspondence $G \rightarrow 1/(4\pi\epsilon_0)$ and the electric quadrupole \rightarrow mass quadrupole substitution, these expressions differ only by a factor of 4, fact which might cast reasonable doubts on gravitational space deformation occurring the same way in any physical situation.

We will indeed show that such an effect just comes from special relativity and that its *parameter free prediction* leads to an alternative simpler interpretation of physical reality.

2 Gravitomagnetic vs. magnetic fields

Our starting point will be the Newton attractive force of two masses M and m at rest (as confirmed by Cavendish experiments). All speculations about the scalar, vector, or tensor origin of this interaction will be for the moment left aside. This interaction is a matter of fact, irrespective of its more or less satisfactory implementation in a Lagrangian language, which is neither a necessary viaticum for truth nor a must for the description of physical reality.

That the magnetic field comes from Special Relativity (SR) transformations of the electric field is well known. Let us briefly recall the pedagogical way this is presented by Feynman [4]. A charged particle q is at rest at a distance r from an infinitely long current carrying wire. The wire is uncharged so that its positive and negative linear charge densities are equal

$$\lambda_+ = \lambda_- = \lambda_0 \quad (3)$$

If we now let the charge q move with velocity v (which of course must not be equal to the drift velocity v_d of the charges of the wire, although in this case the given example requires only elementary calculations of immediate physical interpretation), *invoking the principle of relativity one predicts just from electrostatic a new force* since the Lorentz transformed

$$\lambda'_+ \neq \lambda'_- \quad (4)$$

The term coming from $\lambda'_+ - \lambda'_- \simeq \lambda_0 v^2/c^2$ can be easily interpreted as giving rise to a force due to the magnetic field \mathbf{B} whose “electric nature”, in terms of the current i , is transparent since $B = 1/(4\pi\epsilon_0 c^2) 2\lambda_0 v/r = \mu_0/4\pi(2i/r)$.

In order to extend these elementary considerations to gravity, one can rephrase the previous case in terms of *repulsive* charges of the same sign, by substituting the positive charges of the wire with a fictitious wire of negative charge density,

parallel to the the first one, at the same distance from q but at the opposite side.

Therefore in the gravitational case the same situation will be realized by considering a mass m , initially at rest, with two coplanar parallel wires at the same distance r and at opposite sites, one at rest (2) with mass density λ_2 and the other (1) moving with velocity v , with the same mass (or energy) density λ_1

$$\lambda_1 = \Delta m_1 / \Delta l_1 = \lambda_2 = \Delta m_2 / \Delta l_2 = \lambda \quad (5)$$

Hence no net force on m .

Let us now impart the same velocity v to m . In its rest frame S' wire 1 will be at rest with $\lambda'_1 \neq \lambda_1$, and 2 will be moving with velocity $-v$, hence with $\lambda'_2 \neq \lambda_2$ (see Fig.1).

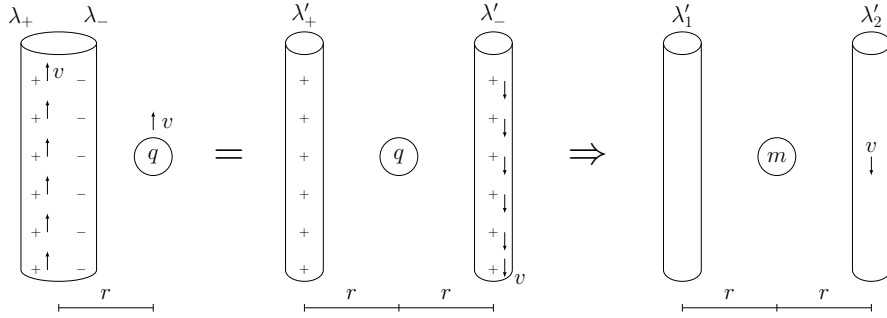


Figure 1: Electromagnetic inspired derivation of gravitomagnetic effects. The neutral straight wire situation, due to opposite charges, is realized in gravitation with a proper placement of another mass wire. The principle of relativity then predicts for the moving mass m an additional velocity dependent “magnetic” force. Relativistic gravitational forces can therefore be also repulsive

Now the obvious but fundamental difference with respect to the *e.m.* case is that not only lengths shrink and stretch, but that also masses vary according to special relativity. Therefore the effect to first order in v^2/c^2 will be a factor of 2 with respect to the *e.m.* case ! This is easy to understand since, just because of dimensional considerations, m and l behave the opposite way with respect to relativistic effects.

A probably not entirely superfluous comment may be that SR would predict no gravitomagnetic force if the effects would subtract ($1-1=0$) instead of adding ($1+1=2$).

From

$$\lambda'_1 = \lambda_1 / \gamma^2(v) = \lambda(1 - v^2/c^2) \quad (6)$$

and

$$\lambda'_2 = \lambda_2 \gamma^2(v) \simeq \lambda(1 + v^2/c^2) \quad (7)$$

it is then immediate to get an “effective” mass density difference m (which is clearly not a four vector because of one factor of γ coming from the mass and the other one from the length)

$$\Delta\lambda' \simeq -2\lambda \gamma^2 (v^2/c^2) \quad (8)$$

which determines

$$g' \simeq \gamma^2 \frac{4G}{c^2} \frac{\lambda v^2}{r} \quad (9)$$

from which one can define, paralleling the e.m. case,

$$\mathbf{h} = \gamma^2 \frac{2G}{c^2} \frac{\mathbf{r} \times \mathbf{j}}{r^2} \quad (10)$$

in terms of the ordinary definition of current density $\mathbf{j} = \rho \mathbf{v} = \frac{\lambda}{S} \mathbf{v}$ (S standing for the area of the wire).

The transformation properties of \mathbf{g} are therefore

$$\mathbf{g}' = \gamma^2 \mathbf{v} \times \mathbf{h} \quad (11)$$

Let us anticipate that the current

$$\mathbf{j}^* = 2\mathbf{j} \quad (12)$$

is the one which satisfies the continuity equation

$$\nabla \cdot \mathbf{j}^* = -\frac{\partial \rho}{\partial t} \quad (13)$$

Loosely speaking, since \mathbf{j}^* and not \mathbf{j} is the quantity which comes from the transformation of ρ , it is the former which combines with the latter to form, at this ($O(v^2/c^2)$) level, the required four vector in the continuity equation. In other words the mass current is *not* obtained from the electric one by the simple minded replacement $q \rightarrow m$! If one is rightly puzzled by this result which seems to clash with the usual current continuity equation, let us reemphasize that this is a v^2/c^2 expansion of a relativistic transformation to get a g' which is then reinterpreted as a current induced magnetic term. Thus, in a sense, this is not the standard $\mathbf{j} = m\mathbf{v}$ one usually deals with.

Along the same lines the same result is recovered for the infinite plane in terms of the surface mass density σ thus confirming in general the existence of a gravitomagnetic field \mathbf{h} .

The consideration of self energy effects, proven to be essential to provide in simple terms for the so called “crucial tests of GR” [7] does not alter our conclusions. Indeed if one uses the correct expression

$$g = 2G \frac{\lambda}{r} (1 - G\lambda/c^2) \quad (14)$$

it is easily seen that the term in round brackets (which disposes of space curvature) yields terms which might be thought of as generating gravitomagnetic

effects of higher order in v^2/c^2 . Its contribution to g' of Eq. (9) is therefore negligible. This should not come as a total surprise also in the traditional formulation of gravity, since vector gravity might represent the most relevant part of the energy momentum tensor. Indeed one can associate our “vectors” to the $O(1/c^2)$ components of the matter energy momentum tensor T_{0j} , whereas the “curvature terms” given by T_{ij} are $O(1/c^4)$.

3 The vector equations

We can thus define the gravitomagnetic force which adds to the Newtonian one as

$$\mathbf{F} = m(\mathbf{g} + \mathbf{v} \times \mathbf{h}) \quad (15)$$

where m is the relativistic mass.

We then see that, as a relativistic effect, gravitation may become repulsive.

Thus a post Newtonian formulation of gravitation has necessarily to embody a short distance repulsion from self energy effects (which modifies Newton’s law) and velocity dependent possibly repulsive terms, both effects, somewhat at variance with the standard picture, coming from elementary considerations.

Of course one might have defined the magnetic field without the factor of 2, which would then enter the gravitomagnetic Lorentz force (in other words it is only the combination of the expression of the gravitomagnetic field \mathbf{h} and the Lorentz force which determines the dynamics). But this in turn is known (at least in e.m. and the analogy seems plausible) to determine the Faraday induction law which reads with our choice

$$\nabla \times \mathbf{g} = -\frac{\partial \mathbf{h}}{\partial t} \quad (16)$$

and which would correspondingly change. Therefore one way or another the gravitomagnetic equations differ from the corresponding Maxwell ones by the factor of 2 required by special relativity, determining at the same time the Lorentz force.

For the above mentioned reasons we prefer to make the role of the modified current explicit. Thus

$$\nabla \times \mathbf{h} = -\frac{4\pi G}{c^2} 2\mathbf{j} \quad (17)$$

The analogy with the e.m. case, apart from the fundamental factor of 2, is plain. We may also parenthetically remark that this parameter free prediction gives a physical justification of Heaviside’s Ansatz [5] which was made, needless to recall, prior to SR.

It is also clear that this equation holds true (like Ampere’s law) only in stationary conditions. In the time dependent case a *gravitational displacement current* has to be introduced in order to satisfy current conservation. The former equation will thus read

$$\nabla \times \mathbf{h} = -\frac{4\pi G}{c^2} 2\mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{g}}{\partial t} \quad (18)$$

where \mathbf{g} represents the “ordinary” Newtonian field obeying

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \quad (19)$$

These two equations are then implemented by the *assumption* (a fortiori even more reasonable than in electromagnetism) of the non existence of a gravitomagnetic charge

$$\nabla \cdot \mathbf{h} = 0 \quad (20)$$

which implies the existence of a gravito vector potential \mathbf{b}

$$\mathbf{h} = \nabla \times \mathbf{b} \quad (21)$$

The implementation, so to say, of SR in Heaviside vector formulation of gravitation [5], thus yields a parameter free set of equations for the weak field, low velocity gravitational case.

All discussions about the necessity of ruling out a vector formulation of GR (although the present one is manifestly an approximate, of the same $O(1/c^4)$ of GR solutions), because it would be repulsive, seem therefore idle.

Since

$$\mathbf{g} = -\nabla\phi - \frac{\partial\mathbf{b}}{\partial t} \quad (22)$$

it is immediate to get for the potentials from the previous equations (ϕ, \mathbf{b}) in the Coulomb-Newton gauge $\nabla \cdot \mathbf{b} = 0$

$$\nabla^2\phi = 4\pi G\rho \quad (23)$$

i.e. *an instantaneous gravitopotential, which thus explains the success of Newton's formulation, and a transverse vector potential*

$$\nabla^2\mathbf{b} - \frac{1}{c^2} \frac{\partial^2\mathbf{b}}{\partial t^2} = -\frac{4\pi G}{c^2} 2\mathbf{j} \quad (24)$$

The particle momentum \mathbf{p} becomes

$$\mathbf{p} = m(\mathbf{p} + \mathbf{b}) \quad (25)$$

in close analogy with the e.m. case. The body m gets an extra contribution to its momentum from the *currents* through \mathbf{b} , in addition to the Newtonian one.

However in our case

$$\mathbf{b} = \frac{G}{c^2} \int \frac{2\mathbf{j}}{|r - r'|_{t'}} dV' \quad (26)$$

It is immediate to see, as a consequence of the previous set of equations, that the propagation velocity of gravity is given by

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = 0 \quad (27)$$

4 Energy balance and radiation

The continuity equation reads

$$\frac{4\pi G}{c^2} \mathbf{j}^* \cdot \mathbf{g} = -(\nabla \times \mathbf{h}) \cdot \mathbf{g} + \frac{1}{c^2} \mathbf{g} \cdot \frac{\partial \mathbf{g}}{\partial t} = \quad (28)$$

$$= -\nabla \cdot (\mathbf{h} \times \mathbf{g}) + \frac{1}{2c^2} \frac{\partial(\mathbf{g}^2 + c^2 \mathbf{h}^2)}{\partial t} \quad (29)$$

and the energy density is given by

$$U = -\frac{1}{4\pi G} \frac{\mathbf{g}^2 + c^2 \mathbf{h}^2}{2} \quad (30)$$

where the gravitoelectric energy density agrees with the corresponding static expression.

The energy flux \mathbf{H} (duely dubbed after Heaviside) is

$$\mathbf{H} = \frac{c^2}{4\pi G} \mathbf{g} \times \mathbf{h} \quad (31)$$

Therefore the energy balance comes out right. Notice also, as a consequence of the former wave equations, that in vacuum

$$h = g/c \quad (32)$$

Therefore in vacuum

$$U = -\frac{1}{4\pi G} g^2 \quad (33)$$

and

$$|H| = \frac{c}{4\pi G} g^2 \quad (34)$$

so that

$$|H| = c U \quad (35)$$

i.e. the usual relation one has between flux and energy density for electromagnetism.

It is easy to see that energy would be conserved *even* with the coefficient 2 in front of the displacement current. This would however imply a propagation velocity $c^* = c/\sqrt{2}$, thus violating causality and affecting radiation where *also the power depends on the propagation velocity*.

Thus we straightforwardly predict a quadrupole radiation

$$W_G = W_{el} \left(\frac{1}{4\pi\epsilon_0} \rightarrow G, q \rightarrow 2m \right) \quad (36)$$

In conclusion the factor of 4 in the quadrupole gravitational radiation, traveling with speed c in a flat Minkowski space, can be simply derived by these elementary considerations based entirely on special relativity.

As a consequence also in all other multipoles the same replacement takes the place of the naive $q \rightarrow m$.

5 Comments and conclusions.

In the present paper the unavoidable contribution of special relativity to gravitation has been calculated.

It has been shown that SR plays a paramount role also in what has been considered so far to be a distinctive feature of gravitation i.e. radiation. This should not come as a total surprise also in the traditional formulation of gravity, since vector gravity might represent the most relevant part of the energy momentum tensor. Indeed one can associate our “vectors” to the $O(1/c^2)$ components of the matter energy momentum tensor T_{0j} , whereas the “curvature terms” given by T_{ij} are $O(1/c^4)$.

Some comments as regards the results obtained and their relation to related works are in order.

Indeed in the literature different versions of the so called GEM (gravitoelectromagnetism) equations are available [10, 6, 11, 12, 13, 14]. All of them are said to be obtained from the basic equation of GR “assuming a weak gravitational field or reasonably flat spacetime” [?]. However the coefficient of the current in the 4th equation varies from 1 to 2 to 4. None of them in addition seems to be interested in the issue dealt with here i.e. the problem of the propagation velocity and the consistent evaluation of the ensuing gravitational radiation. All of them are incomplete and/or wrong (remember the previous comments about the necessarily constrained form of the Lorentz force and of the induction law).

In the case the current coefficient is 1, which corresponds to the naive Heaviside case, the propagation velocity is clearly c , but the quadrupole radiation is 1/4 of the one predicted by GR. Therefore the reduction violates both GR and SR, since the relativistic mass variation implies the coefficient to be 2!

If the coefficient is 2, as has been derived in the present work from SR, and the same factor multiplies the displacement current the propagation velocity would be $c/\sqrt{2}$ which manifestly violates causality. In addition one would predict a bigger quadrupole radiation by the same amount $\sqrt{2}$.

If the coefficient is 4 (the extra factor of 2 being claimed to come from spacetime distortion) the propagation velocity is $c/2$ and quadrupole radiation would be 16 times bigger than predicted by GR!

In conclusion both the expressions for the periastron precession

$$\frac{\Delta\phi}{\phi} = 3G \frac{M_1 + M_2}{c^2 a (1 - e^2)} \quad (37)$$

and

$$W_Q = 32G^4 \frac{(M_1 M_2)^2 (M_1 + M_2)}{5c^5 a^5 (1 - e^2)^{7/2}} \times \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right) \quad (38)$$

based on the quoted expression for quadrupole radiation, basis for the PN description of the periastron advance and period T decrease of binary systems (Shapiro effects deriving trivially from space-time “curvature”) are easily derived from elementary considerations. The first one from self energy which affects Newton’s law and angular momentum [7].

In this connection it is instructive to recall the beautiful example by Feynman of the bugs on a hot plate. If they unaware of the temperature gradient they will conclude that they live in a curved (non euclidean) space, which will on the contrary appear flat to us who know of the temperature effects. In the same way if we ignore self energy effects, we will attribute to space time curvature our ignorance. The two physical descriptions of our world are of course equivalent, apart from physical foundedness, simplicity and from the problem of energy and momentum conservation in a curved space time.

Self energy effects have also been seen to play a paramount role in an alternative consistent cosmological description [8].

In conclusion the endeavour to derive GR from SR started by Schiff [9] and pursued in [7] for Mercury precession has come to an end with the present work.

The factor of 2 in the current and therefore in the quadrupole is thus seen to come from an (almost) elementary twofold consequence of Lorentz transformations and not *necessarily* from a hypothetical spin 2 nature of the graviton.

The prediction of the geodetic precession and of frame dragging as a particular, i.e. stationary, case of the present equations is being considered elsewhere.

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